

Slow convergence tunes onset of strongly discontinuous explosive percolation

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Contrary to initial beliefs, random graph evolution under an edge competition process with fixed choice (an Achlioptas process) seems to lead to a continuous transition in the thermodynamic limit. Here we show that a simpler model, which examines a single edge at a time, can lead to a strongly discontinuous transition and we derive the underlying mechanism. Starting from a collection of n isolated nodes, potential edges chosen uniformly at random from the complete graph are examined one at a time while a cap, k , on the maximum allowed component size is enforced. Edges whose addition would exceed size k can be simply rejected provided the accepted fraction of edges never becomes smaller than a decreasing function, $g(k) = 1/2 + (2k)^{-\beta}$. If the rate of decay is sufficiently small ($\beta < 1$), troublesome edges can always be rejected, and the growth in the largest component is dominated by an overtaking mechanism leading to a strongly discontinuous transition. If $\beta > 1$, once the largest component reaches size $n^{1/\beta}$, troublesome edges must often be accepted, leading to direct growth dominated by stochastic fluctuations and a “weakly” discontinuous transition.

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Percolation is a theoretical underpinning for analyzing properties of networks, including epidemic thresholds, vulnerability and robustness [1–6], with large-scale connectivity typically emerging in a smooth and continuous transition. A new class of percolation was conjectured to exist two years ago, when it was shown that including a small amount of fixed choice in what to connect up leads to an abrupt and seemingly discontinuous “Explosive Percolation” transition [7]. In just the past months it has been established that any random graph evolution process with fixed choice in fact leads to a continuous transition in the thermodynamic limit [8–11]. Here we show that a simple stochastic process can lead to a discontinuous transition and we analytically derive the underlying mechanism. Starting from a collection of isolated nodes, potential edges chosen uniformly at random are examined one at a time while a cap on maximum allowed component size is enforced. Undesirable edges are simply rejected provided that the accepted fraction of edges is never smaller than a decreasing function. We show that if this function decreases sufficiently slowly then undesirable edges can always be rejected leading to growth via overtaking and a strongly discontinuous percolation transition.

A prototypical process of percolation on a network begins from a collection of n isolated nodes with edges connecting pairs of nodes sequentially chosen uniformly at random and added to the graph [12]. In a process with fixed choice, as studied in [7], instead, a fixed number of randomly selected candidate edges are examined together, but only the edge that maximizes or minimizes a pre-set criteria is added to the graph. The rule in [7] min-

imizes the product of the sizes of the two components to be merged. Numerical evidence indicates this rule leads to an abrupt jump in the fraction of nodes in the largest component at the critical point due to the addition of one single edge. Since the publication of [7] several related processes have been shown to result in similar transitions [13–19], yet, it has also been observed that the size of the jump at the critical point decreases with system size [8, 18, 20] (termed “weakly” discontinuous), with the most recent results showing that fixed choice percolation transitions on networks are in fact *continuous*, with no jump, in the thermodynamic $n \rightarrow \infty$ limit [8–11]. (Yet, fixed choice processes on lattices may be discontinuous [21].)

The basic model we analyze here was originally introduced by Bohman, Frieze and Wormald (BFW) [22], and predates the work on explosive percolation. The BFW process is initialized with a collection of n isolated nodes and a cap on the maximum allowed component size set to $k = 2$. Edges are then sampled one at a time, uniformly at random from the complete graph. If an edge would lead to formation of a component of size less than or equal to k it is accepted. Otherwise the edge is rejected provided that the fraction of accepted edges remains greater than or equal to a function $g(k) = 1/2 + (2k)^{-1/2}$. If the accepted fraction would not be sufficiently large, the cap is augmented to $k + 1$ repeatedly until either the edge can be accommodated or $g(k)$ decreases sufficiently that the edge can be rejected. (For the explicit BFW algorithm see the Supplementary information (SI).)

Here we modify the original function such that $g(k) = \min(1, 1/2 + (2k)^{-\beta})$. The parameter β controls the rate of convergence to the asymptotic limiting value of $1/2$, as shown in Fig. 1(a). Letting C_i denote the fraction of nodes in the i th largest component, we show both analytically and via numerical investigation that for $\beta < 1$ any significant growth in C_1 is dominated by an overtaking

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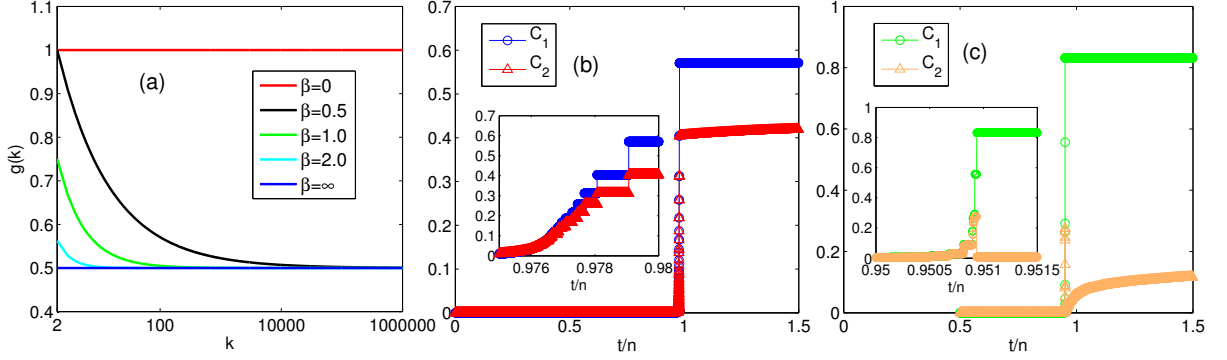


FIG. 1: Convergence rate determines growth mechanism. (a) The function $g(k)$ for different β . Note, $g(k) = 1$ means all edges must be accepted equivalent to Erdős-Rényi [12]. (b-c) Evolution of C_1 and C_2 versus edge density, t/n for $n = 10^6$. (b) For $\beta = 1/2$, two giant components emerge simultaneously. Inset is the behavior in the critical region showing growth via the overtaking process when what was C_1 becomes C_2 . (c) A typical realization for $\beta = 2.0$. Inset shows direct growth, that C_1 and C_2 merge together, and what was C_3 becomes the new C_2 .

mechanism where smaller components merge together to become the new largest component, leading to an evolution that converges to a well-defined probability distribution and a strongly discontinuous transition. In contrast, if $\beta > 1$ significant direct growth of C_1 is allowed and the process is dominated by stochastic fluctuations, leading to a weakly discontinuous transition that is likely continuous as $n \rightarrow \infty$. The typical evolution of C_1 and C_2 for $\beta = 0.5$ and $\beta = 2.0$ are shown in Figs. 1(b) and (c) respectively. (The simultaneous emergence of multiple stable giant components was recently shown in [23], but the underlying mechanism leading to the discontinuous transition, our current focus, was not identified.)

We numerically measure C_1, C_2 , and C_3 throughout the evolution process for various $\beta \in [0.5, \infty]$, for a large ensemble of realizations and range of system sizes n . For each realization we define the critical point as the single edge t_c whose addition causes the largest change in the value of C_1 , with this largest change denoted by ΔC_{\max}^1 . As shown in Fig. 2(a), for $\beta = 0.5$, ΔC_{\max}^1 is independent of system size n and strongly discontinuous. The same holds for ΔC_{\max}^2 , the largest jump in C_2 . For $\beta = 2.0$, in contrast, the transition is weakly discontinuous meaning that ΔC_{\max}^1 decreases with n .

As discussed in [20], whenever a single edge is added to the evolving graph, C_1 may increase due to one of three mechanisms: (1) **Direct growth**, when the largest component merges with a smaller one; (2) **Doubling**, when two components both of fractional size C_1 merge (this is the largest increase possible); (3) **Overtaking**, when two smaller components merge together to become the new largest. In [20] it is strictly proven that if direct growth is prohibited up to the step when only two components are left in the system, then a strongly discontinuous transition ensues. We next show via analytic arguments that throughout the subcritical regime, for $\beta < 1$, direct growth only occurs when the largest component merges with an essentially isolated node and all signifi-

cant growth is due to overtaking. In contrast, for $\beta > 1$ the evolution is initially the same, but once $C_1 \sim n^{1/\beta}$, large direct growth of C_1 dominates.

Using the notation of [22], let t denote the number of accepted edges and u the total number of sampled edges. Thus for any k (the maximum allowed component size), the fraction of accepted edges $t/u \geq g(k)$. If t/u is sufficiently large an edge leading to $C_1 n > k$ can be simply rejected. In contrast, if $t/u = g(k)$ and the next edge sampled, denoted e_{u+1} , would lead to $C_1 n > k$ that edge cannot be simply rejected since $t/(u+1) < g(k)$. One of two situations must happen, either: (i) k increases until the edge e_{u+1} is accommodated, or (ii) $g(k)$ decreases sufficiently that $t/(u+1) \geq g(k)$ and e_{u+1} is rejected. Thus we need to determine the order of the smallest augmentation of k that makes $t/(u+1) > g(k)$. For $\beta \geq 0.5$ and any k the smallest fraction of accepted edges is

$$\frac{t}{u} = g(k) = \frac{1}{2} + \left(\frac{1}{2k}\right)^\beta. \quad (1)$$

Rearranging Eq. 1 and differentiating by k yields

$$\frac{du}{dk} = \frac{\beta}{2^\beta [1/2 + (1/2k)^\beta]^2} \frac{t}{k^{\beta+1}} \quad (2)$$

At some point in the subcritical regime we know that $t \sim \mathcal{O}(n)$. For $\beta = 0.5$ this has been established rigorously, specifically that by the end of stage $k = 25$, $t/n \rightarrow 0.841$ as $n \rightarrow \infty$ [22]. For other β , we establish this via numerical simulation (see SI, Fig S3(a)). With this in place we find that once $k \sim n^\gamma$ (with $\gamma < 1$ for the subcritical region) then $du/dk \sim n^{1-\gamma\beta-\gamma}$. Thus an increase in k of order $\mathcal{O}(n^{\gamma\beta+\gamma-1})$ is sufficient to ensure $t/(u+1) \geq g(k)$ and that edge e_{u+1} can be rejected. But there are different behaviors for $\beta > 1$ and $\beta < 1$.

For $\beta > 1$ there are three regimes. For $\gamma \leq 1/(\beta+1)$ then $\gamma\beta+\gamma-1 \leq 0$ so the necessary $\mathcal{O}(n^{\gamma\beta+\gamma-1})$ increase in k requires only $k \rightarrow k+1$. Then once $1/(\beta+1) < \gamma <$

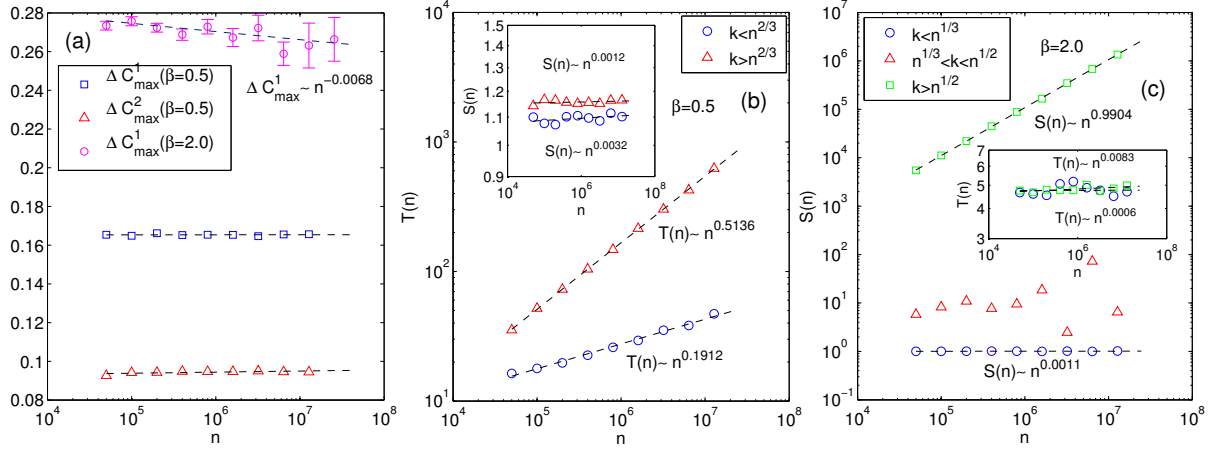


FIG. 2: Slow convergence leads to a strongly discontinuous transition and growth by overtaking. (a) For $\beta = 0.5$, ΔC_{\max}^1 and ΔC_{\max}^2 are independent of n and both largest components emerge in strongly discontinuous phase transitions. For $\beta = 2.0$, $\Delta C_{\max}^1 \sim n^{-0.0068}$, showing a weakly discontinuous phase transition. (b) For $\beta = 0.5$, main plot is $T(n)$ the number of times C_1 undergoes **direct growth** versus n , with two regimes separated by $k = n^{1/(\beta+1)}$. (Inset) $S(n)$, the average size of component that merges with C_1 during direct growth, is essentially a constant: $S(n) \approx 1.1$ in both regimes. (c) For $\beta = 2.0$, main plot is $S(n)$, showing three regimes (see SI, Fig. S1 for $\beta = 3$). For $k < n^{1/(\beta+1)}$ we observe $S(n) \approx 1$. The intermediate regime is noisy. Then once $k > n^{1/3}$, random edges must be accepted at times and $S(n) \sim n^{0.9904}$ (C_1 merges with other essentially macroscopic components). (Inset) $T(n)$ is essentially constant: $T(n) \approx 5$. All data points are the average over 30 to 100 independent realizations (based on system size), with error bars smaller than the symbols unless otherwise indicated.

$1/\beta$, an increase in k of $\mathcal{O}(n^{\gamma\beta+\gamma-1}) < k \sim C_1 n$ is required. However, once $\gamma > 1/\beta$, then $\mathcal{O}(n^{\gamma\beta+\gamma-1}) > C_1 n$. The required increase in k is greater than $C_1 n$, allowing C_1 to even double in size, and edge e_{u+1} must be accepted. So, once in the regime $C_1 n \sim n^{1/\beta}$ every time $t/u = g(k)$ the next edge, e_{u+1} , must be accepted. In this situation, the probability two components are merged becomes, as in Erdős-Rényi [12], proportional to the product of their sizes and no mechanism remains enforcing that the components stay of similar sizes.

For $\beta \in [0.5, 1)$ there are only two regimes. Here again for $\gamma < 1/(\beta+1)$ then $k \rightarrow k+1$ allows edge e_{u+1} to be rejected. This regime extends until $\gamma \geq 1/(\beta+1)$, when $\gamma\beta+\gamma-1 \geq 0$, but now we use the less strict property that $\beta\gamma+\gamma-1 \leq \beta$ and thus $n^{\beta\gamma+\gamma-1} \leq n^\beta < k$. So k increases in increments of n^β at most, and edge e_{u+1} is rejected. The slow increase results in multiple components of size similar to C_1 throughout the evolution. In particular, for $C_1 = \delta n$ with $\delta \ll 1$, there exist many components of size $\mathcal{O}(n)$. Order them as $C_1 n \geq C_2 n \geq \dots \geq C_l n$. Assuming “ $>$ ” strictly holds (i.e., choosing non-degenerate components), due to the $\mathcal{O}(n^\beta)$ increase in k , there will be components such that $C_l + C_{l-1} > C_1$ (or else two smallest components of size $\mathcal{O}(n)$ would merge very quickly) and there will be a point when $(C_l + C_{l-1})n < k < (C_l + C_{l-2})n$, allowing for growth by overtaking. We explicitly observe this overtaking process for $\beta = 0.5$ until only three components of size $\mathcal{O}(n)$ remain. Then we add edge t_c and C_2 and C_3 merge for the final overtaking event as shown inset in Fig. 1(b).

We confirm these predictions via numerical simulations

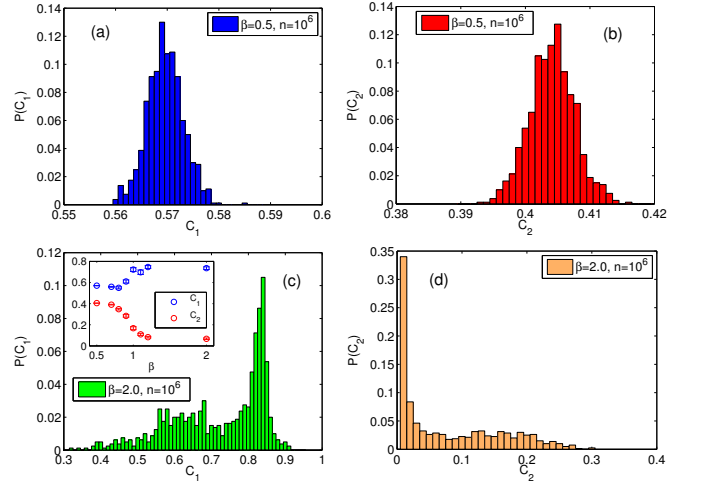


FIG. 3: (a-d) Distribution in values of C_1 and C_2 at t_c obtained over 100 independent realizations for $n = 10^6$. (t_c is defined as the single edge whose addition causes the biggest increase in C_1 .) (a) C_1 for $\beta = 0.5$. (b) C_2 for $\beta = 0.5$. (c) C_1 for $\beta = 2.0$. (d) C_2 for $\beta = 2.0$. Inset to (c) shows average size of C_1 and C_2 at t_c over 100 realizations for different β .

using two choices, $\beta = 1/2$ and $\beta = 2.0$. Let $S(n)$ denote the average size of the component C_i which connects to C_1 via direct growth for a system of size n , and let $T(n)$ denote the number of times direct growth occurs. Figure 2(b) is for $\beta = 1/2$ for the two regimes which are separated by $k = n^{1/(\beta+1)} = n^{2/3}$. As shown in the inset, throughout both regimes $S(n) \approx 1.1$ is essentially a positive constant. But $T(n)$ (the main figure) shows a distinct regime change. At first $T(n) \sim n^{0.19}$. Then in

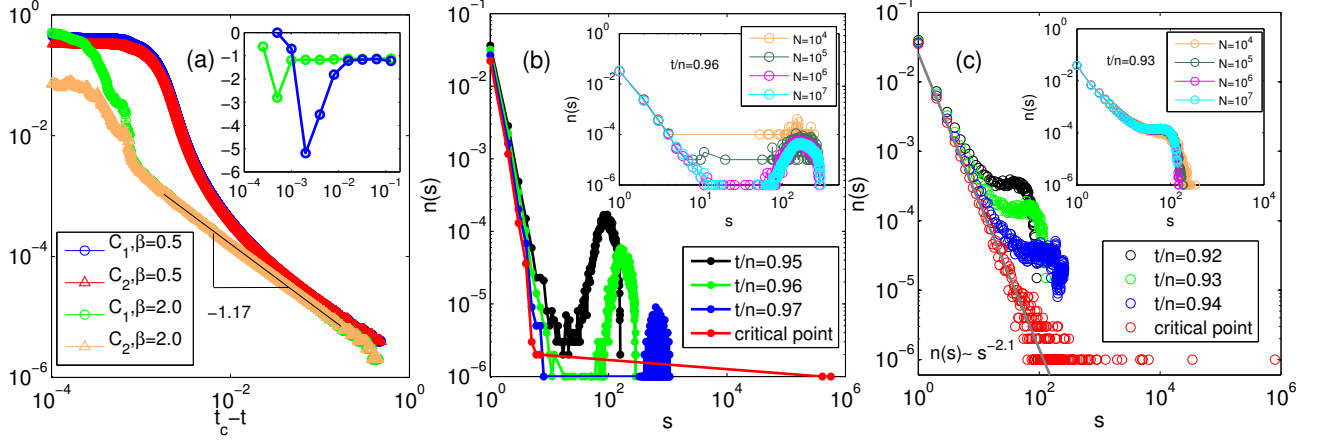


FIG. 4: No evident scaling behaviors for $\beta = 0.5$ whereas quantities for $\beta = 2.0$ exhibit critical scaling. (a) C_1 and C_2 versus $t_c - t$. For $\beta = 2.0$ both $C_1, C_2 \sim (t_c - t)^{-1.17}$, yet $\beta = 0.5$ shows no obvious scaling. Inset is the local slope estimate for C_1 . (b) Distribution of component density $n(s)$ (number of components of size s divided by n) at different points in the evolution for $\beta = 0.5$. Inset is $n(s)$ at $t/n = 0.96$ for various n , showing no finite size effects in the location of the right hump. (c) Evolution of $n(s)$ for $\beta = 2.0$, with $n(s) \sim s^{-2.1}$ at t_c . Inset is $n(s)$ at $t/n = 0.93$ for various n , again showing no finite size effects.

the regime starting with $k = n^{2/3}$ up to and including t_c we see a much more rapid increase $T(n) \sim n^{0.51}$. So we see direct growth occurring more frequently in the second regime, but the direct growth continues to be due to merging with an essentially isolated node.

Figure 2(c) is for $\beta = 2$, with $S(n)$ for the three regimes shown in the main plot. Up until $k = n^{1/(\beta+1)} = n^{1/3}$ the behavior is the same as for $\beta = 0.5$ as expected since $\Delta k = 1$ is enough for an edge to be rejected and we see $S(n) \approx 1.1$. Then in the regime $n^{1/3} < k < n^{1/2}$, $S(n)$ is larger and has large fluctuations. Finally in the regime from $k = n^{1/\beta} = n^{1/2}$ up until edge t_c , we see C_1 grow in large bursts, with $S(n) \sim n^{0.99}$, so C_1 merges with other essentially macroscopic components. (Fig. S1 of the SI shows equivalent behavior for $\beta = 3.0$.) As shown inset, $T(n)$ is essentially independent of regimes, with $T(n) \approx 5$ in the first regime and in the third, with $T(n) \leq 1$ but fluctuating in the second (not shown).

For $\beta = 2.0$, due to the acceptance of random edges once $k > n^{1/2}$, we observe large fluctuations in the size of the giant components at t_c . And due to the slow increase in k for $\beta = 0.5$, the sizes of the components evolve in a predictable manner. Figure 3 shows these behaviors, with (a) and (b) C_1 and C_2 observed at t_c over 100 independent realizations for $\beta = 0.5$ and (c) and (d) the equivalent for $\beta = 2$. Our theoretical analysis holds for all $\beta < 1$, yet, with numerical analysis we could only confirm these behaviors for $\beta \in [0.5, 0.7]$, which may be due to finite size effects.

We also carry out a scaling analysis on the general BFW model. We first consider the behavior of both C_1 and C_2 near t_c , as shown in Fig. 4(a). For $\beta = 2$ we find traditional critical scaling, that $C_1, C_2 \sim (t_c - t)^{-\eta}$ with $\eta = 1.17$, the same scaling as displayed by the Product Rule (PR) edge competition rule studied in [7] [17, 24]. For $\beta = 0.5$, C_1, C_2 show no obvious scaling behavior.

More importantly we study the component size density $n(s)$ (the number of components of size s divided by n). We measure the distribution of $n(s)$ at different points in the evolution up to the critical point. Note for $\beta = 0.5$, $t_c \approx 0.976n$ and for $\beta = 2.0$, $t_c \approx 0.951n$ (see SI, Fig. S4(a)). For $\beta = 2.0$, Fig. 4(c), the behavior is similar to that for PR and other edge competition models with fixed choice, where at the critical point there is clear scaling behavior, $n(s) \sim s^{-\tau}$ with $\tau = 2.1$ (the same τ as for PR). Yet, as shown in Fig. 4(b), the evolution for $\beta = 0.5$ does not show any scaling. There is a pronounced right-hump which forms early in the evolution, then moves rightward due to overtaking. Inset to Fig. 4(b) and (c) is $n(s)$ at $t = 0.96$ and $t = 0.93$ respectively, for many different n . Note that the peak of the right-hump is independent of n . This is quite distinct from Erdős-Rényi and PR which show finite size effects; for PR the location of the peak moves rightward with n , as recently shown in [11] where a finite size scaling function for PR is established.

Also interesting is the rank-size component distribution at t_c . C_i for $\beta = 0.5$ does not show scaling, yet for $\beta = 2$ we find $C_i \sim i^{-\delta}$ with $\delta = 0.62$ (see SI, Fig. S4(d)). (Note the $\beta = 2$ model and PR have the same η and τ exponents, but for PR $\delta = 0.90$.) Finally we define N_k as number of distinct stages visited up to time t_c . We find that $N_k \sim n^{0.603}$ for $\beta = 0.5$ and $N_k \sim n^{0.292}$ for $\beta = 2.0$ (see SI, Fig. S4(c)). More stages are visited for $\beta = 0.5$ consistent with our theoretical analysis.

In summary, we have shown that the convergence rate of the generalized BFW model tunes the onset of discontinuous percolation. For sufficiently slow convergence, troublesome edges can always be rejected and significant growth in C_1 occurs only via overtaking, leading to the discontinuous emergence of multiple giant components. Algorithms that generate multiple giant components may

have a range of applications, such as controlling gel sizes during polymerization [25] or creating building blocks for modular networks. In addition, the BFW model may be simpler to implement in practice than fixed-choice edge competition models. A more rigorous understanding of the evolution of $n(s)$ for $\beta < 1$ as $n \rightarrow \infty$ would be useful. Of course real-world networks such as social networks or the Internet, are by definition of finite size, so the implications of asymptotic theory may not apply. For instance, for the fixed-choice edge competition model in [8] it was shown that systems as large as 10^{18} nodes still exhibit a large macroscopic jump of 10% of the

system size. (PR decays more quickly. Using the scaling reported in [18, 20, 23] for systems of 10^{18} the jump would be approximately 3%, for 10^9 it would be 10%.) Finally, the Gaussian model [19] which also only controls the largest cluster size also appears to be strongly discontinuous [26].

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Supplementary Information:

Slow convergence tunes onset of strongly discontinuous explosive percolation

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In the main manuscript we study the BFW model with the generalized acceptance function $g(k) = \min(1, 1/2 + (2k)^{-\beta})$. We show that if $\beta > 1$ eventually in the subcritical regime once, $k \sim n^{1/\beta}$, there are times that an undesirable random edge must be accepted. In contrast, if $\beta < 1$ undesirable edges can always be rejected, leading to growth by overtaking and a strongly discontinuous phase transition. Here we present additional technical details that are not necessary for the main manuscript but either highlight interesting aspects of the BFW model or provide further clarification.

S.1 The BFW algorithm

Stating the BFW model explicitly, it begins with a collection of n isolated nodes and proceeds in phases starting with $k = 2$. Edges are sampled one at a time, uniformly at random from the complete graph induced on the n nodes. Using the notation in [2], let u denote the total number of edges sampled, A the set of accepted edges (initially $A = \emptyset$), and $t = |A|$ the number of accepted edges. At each step u , the selected edge e_u is examined via the following algorithm (the BFW algorithm):

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Set  $l = \text{maximum size component in } A \cup \{e_u\}$ 
  if  $(l \leq k) \{$ 
     $A \leftarrow A \cup \{e_u\}$ 
     $u \leftarrow u + 1 \}$ 
  else if  $(t/u < g(k)) \{ k \leftarrow k + 1 \}$ 
  else  $\{ u \leftarrow u + 1 \}$ 

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Walking through the loop, if adding e_u leads to a component of size $l \leq k$, the edge is added and step u completes and we move onto step $u + 1$. Otherwise, check if $t/u < g(k)$. If not, then $t/u \geq g(k)$ and e_u is ignored (*i.e.* rejected) and we move onto step $u + 1$. If instead it is the case that $t/u < g(k)$ then we augment k and walk through from the “if” loop from the start again. In other words, while $t/u < g(k)$, k is augmented by one repeatedly until either k becomes large enough that edge e_u is accepted or $g(k)$ decreases sufficiently that edge e_u can be rejected at which point step u finally ends. Note $g(k) = 1$ requires that all edges be accepted, equivalent to Erdős-Rényi [1]. Also we use the “min” function, $g(k) = \min(1, 1/2 + (2k)^{-\beta})$, to ensure $g(k) \leq 1$ for all choices of β , since the fraction of nodes accepted cannot exceed unity.

T. Bohman, A. Frieze, and N. C. Wormald established rigorous results whereby setting $g(200) = 1/2$, all components are no larger than $k = 200$ nodes (*i.e.*, no giant component exists) when $m = 0.96689n$ edges out of $2m$ sequentially sampled random edges have been added to graph [2]. They further establish that a giant component must exist by the time $m = c^*n$ out of $2m$ sampled edges have been added, with $c^* \in [0.9792, 0.9793]$. Yet, they did not analyze the nature of the percolation transition.

In [3] it was shown that the original BFW model leads to the simultaneous emergence of two giant components (each with fractional size $C > 0.4$), and the stability of the multiple giants was analyzed. (Essentially, once in the supercritical regime, there are always sufficient edges internal to components sampled that whenever an edge connecting two giant components is sampled it can be rejected.) It was also shown in [3] that by using the function $g(k) = \alpha + (2k)^{-1/2}$, then the parameter α tunes the number of stable giants that emerge simultaneously.

S.2 Large fluctuations for $\beta > 1$

As discussed in the main manuscript, for $\beta > 1$ once $k \sim n^{1/\beta}$, then there are situations in which an edge must be accepted, regardless of the size of the resulting component. (Figure S1 shows this for $\beta = 3$.) This leads to large fluctuations in the size of the resulting component, and in the sizes of the components at t_c as shown in Fig. 3 (c) and (d) of the main manuscript. The typical scenario is as illustrated in Fig. 1(c) of the main manuscript, that $C_1 \approx 0.8$ and C_2 emerges in a continuous transition starting from $C_2 = 0$. Yet, as seen in Fig. 3 (c) and (d), we also observe scenarios where C_1 is much smaller and C_2 is already of macroscopic size at t_c . Figure S2(a) shows such a scenario, where $C_1 \approx 0.68$ and $C_2 \approx 0.12$ at t_c , with the inset showing that C_1 still emerges via direct growth. The situation for $\beta = 1/2$, instead is simpler. C_1 and C_2 have a well defined probability distribution at t_c , as shown in the main manuscript in Fig. 3 (a) and (b), where $C_1 = 0.570 \pm 0.001$ and $C_2 = 0.405 \pm 0.001$. (Note the error bars were obtained in Ref. [3]). Figure 1(b) illustrates this only scenario, with the inset showing that C_1 emerges due to overtaking.

We numerically study the behavior for a few more choices of β . The inset to Fig. 3 (c) in the main manuscript shows the average value and corresponding error bars for C_1 and C_2 over 100 different network realizations at t_c for a range of values of β . Here, Fig. S2(b) shows the corresponding standard deviation in the size of these two largest components, denoted as χ_1, χ_2 respectively, over a larger ensemble of 800 realizations. We also numerically measure the number of times the two largest components ever merge together up to and including edge t_c for different choices of β , as show in Fig. S2(c). The interval $\beta \in (0.7, 1)$ separates the region where growth by overtaking dominates from the region where direct growth dominates. Although our analysis in the main manuscript holds for $\beta < 1$, we could only confirm the behaviors numerically for $\beta \in [0.5, 0.7]$, which we suspect is due to finite size effect once $\beta > 0.7$, when occasionally C_1 and C_2 merge.

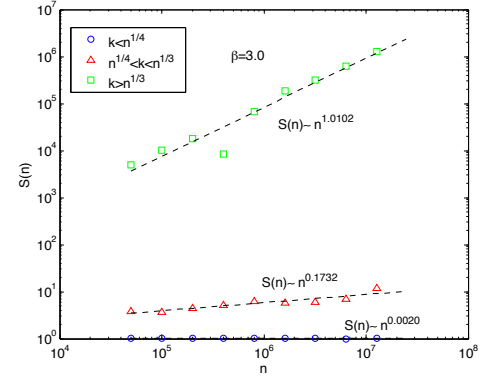


Figure S1: $S(n)$ for $\beta = 3$, analogous to Fig. 2(c) in the main, showing C_1 merges with macroscopic components once $k > n^{1/\beta}$.

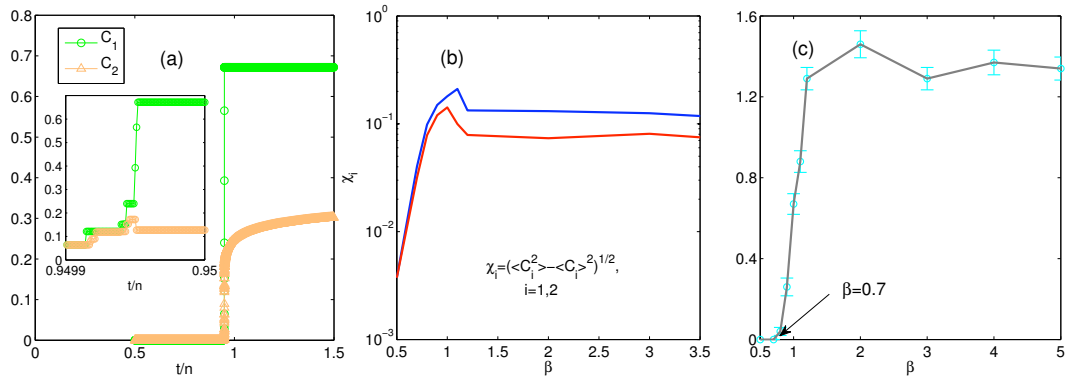


Figure S2: System size here is $n = 10^6$. (a) The sizes of C_1 and C_2 at t_c exhibit large fluctuations for $\beta = 2.0$ (Fig 3 of main manuscript). Typically $C_2 = 0$, but there are also significant number of instances where C_2 is already of macroscopic size at t_c , with a typical such realization shown here. Inset shows that, regardless, C_1 is augmented via direct growth with C_2 , and what was the third largest component (which has macroscopic size) becomes the new C_2 . (b) Standard deviation of C_1 (blue line) and C_2 (red line) at t_c obtained over 800 independent realizations for each β , showing a sharp peak at $\beta \approx 1$. (c) The total number of times that C_1 and C_2 ever merge together measured over 100 realization, showing that this never happens for $\beta < 0.7$.

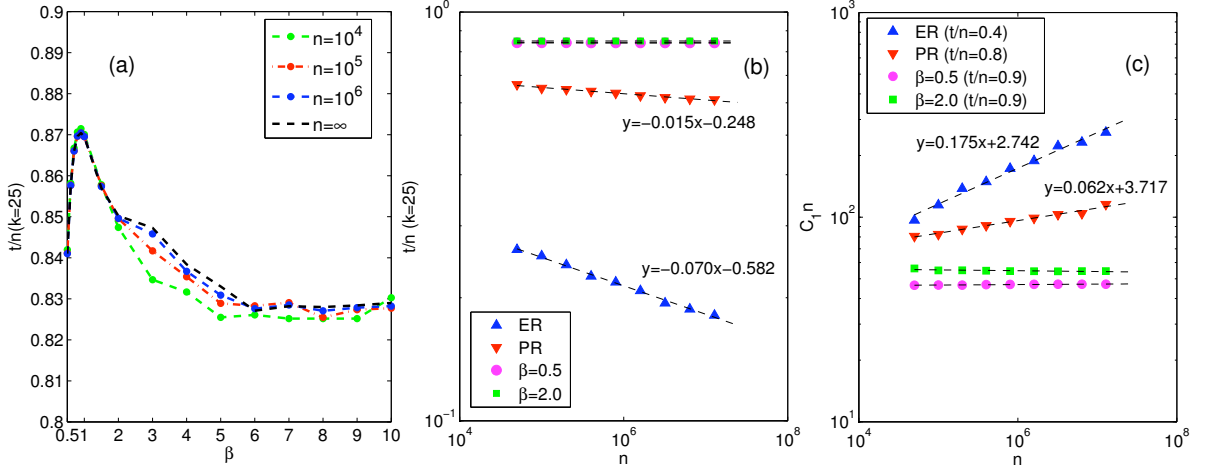


Figure S3: (a) Fraction of added edges, t/n , by the end of stage $k = 25$ versus β for different system sizes showing a peak at $\beta \approx 1$. Values marked $n = \infty$ were obtained from linear extrapolation. (b) Fraction of added edges, t/n , by the end of stage $k = 25$ versus different system sizes for ER, PR, and BFW with $\beta = 0.5$ and $\beta = 2.0$. (c) Size of largest component versus different system size at the time when a specified fraction of edges have been added for ER, PR, and BFW with $\beta = 0.5$ and $\beta = 2.0$.

S.3 Establishing $t \sim \mathcal{O}(n)$ and lack of finite size effects

To analyze Eqn. 2 in the main manuscript we must establish that $t \sim \mathcal{O}(n)$ in the subcritical regime. In [2] they establish this rigorously for $\beta = 0.5$, specifically that $t/n \rightarrow 0.841$ as $n \rightarrow \infty$ by the end of stage $k = 25$. Here we establish this fact for a range of β and at the same time show the lack of finite size effects for the BFW model. Let $\frac{t}{n}(k)$ be the fraction of edges added by the end of stage k . We measure $\frac{t}{n}(k = 25)$ for a series of values of β with $0.5 \leq \beta \leq 10$ for different system sizes, as shown in Fig. S3(a). (Each data point is the average over 100 independent realizations.) From these measurements for different n we obtain the asymptotic $n \rightarrow \infty$ value of $\frac{t}{n}(k = 25)$ by linear extrapolation, showing t/n asymptotically converges to positive constant for all $\beta \geq 0.5$ and the number of added edges $t \sim \mathcal{O}(n)$ at stage $k > 25$. Figure S3(b) shows that $\frac{t}{n}(k = 25)$ is essentially independent of system size and converges to a positive constant for BFW for both $\beta = 0.5, 2.0$. However, $\frac{t}{n}(k = 25)$ decreases to 0 asymptotically for ER and PR, with $\frac{t}{n}(k = 25) \sim n^{-\tau}$, $\tau = 0.070$ and 0.015 respectively. (Note PR is the Product Rule graph evolution process studied in [4].)

On the other hand, rather than measuring t/n for fixed k , we can measure the value of k (which is essentially equal to C_1) at the time when t/n attains a specified value. Figure S3(c) shows that for BFW with $\beta = 0.5, 2.0$, $k \approx C_1 n$ is essentially a positive constant for $t/n = 0.9$. Whereas for ER and PR then $C_1 n \sim n^\theta$ with $\theta = 0.175, 0.062$ respectively. As also seen in Fig. 4(b) and (c) of the main manuscript, the component size density distribution $n(s)$ has a right-side hump for the BFW model (both for $\beta = 0.5, 2.0$) whose peak is independent of system size. In contrast, for PR, the location of the peak (which only exists in the subcritical region) increases with system size [5, 6].

S.4 Scaling behaviors and bounding t_c

We again focus on BFW model with parameter choices $\beta = 0.5$ and 2.0 . To bound the critical window for the transitions we implement the numerical method proposed in [4]. We define $t_0(n)$ as the maximal number of edges for which the largest component has size at most n^γ and $t_1(n)$ as the minimal number of edges for which the largest component has size larger than An where n is system size, and γ and A are

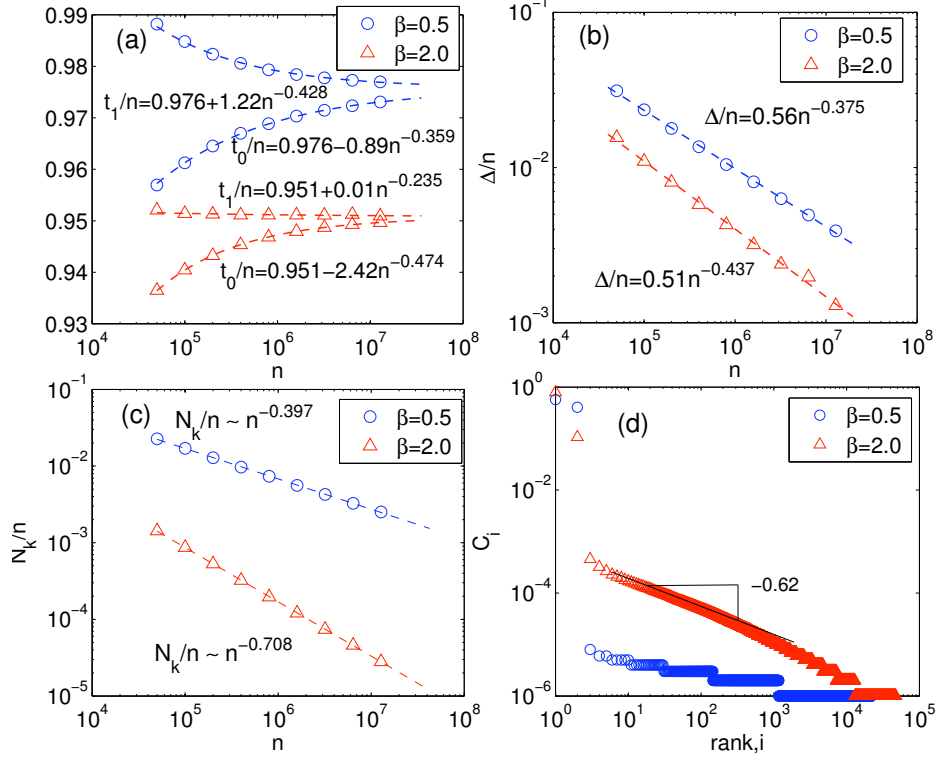


Figure S4: (a) The critical window as bounded by t_0/n and t_1/n showing convergence in first three significant figures to 0.976 and 0.951 for $\beta = 0.5$ and 2.0 respectively. (b) Δ/n the fractional number of edges added between t_0 and t_1 is sublinear in n for both $\beta = 0.5, 2.0$, but decays more rapidly for $\beta = 2.0$ showing a tighter scaling window. (c) N_k/n , the number of distinct values of k visited up to time t_c divided by system size n , shows scaling behaviors for both $\beta = 0.5$ and 2.0, with more distinct stages visited for $\beta = 0.5$. (d) Rank-size component distribution for $\beta = 0.5, 2.0$ at t_c . For $\beta = 2.0$, fitting for $10 < i < 1000$ yields $C_i \sim i^{-\delta}$ with $\delta = 0.62$. But no scaling can be discerned for $\beta = 0.5$.

two parameters. $\Delta(\gamma, A) = t_1(n) - t_0(n)$ denotes the number of edges added between these two points. We set $\gamma = 1/2, A = 0.5, 0.55$ for $\beta = 0.5, 2.0$ respectively. Fig. S4(a) shows that, for the first three significant figures, both t_0/n and t_1/n approach the same limiting value $\lim_{n \rightarrow \infty} t_c/n = 0.976$ for $\beta = 0.5$ and $\lim_{n \rightarrow \infty} t_c/n = 0.951$ for $\beta = 2.0$. We also numerically measure that $\Delta(1/2, 0.5)/n \sim n^{-0.375}$ for $\beta = 0.5$ and $\Delta(1/2, 0.55)/n \sim n^{-0.437}$ for $\beta = 2.0$, which is shown in Fig. S4(b). The model with $\beta = 2.0$ results in a tighter scaling window because in the subcritical region the direct growth mechanism allows the largest component to connect to other large components, thus the largest component can grow very quickly in size. The same reason can be applied to explain why this evolution process shows a more delayed onset of giant component for $\beta = 0.5$. Although Figs. S4(a) and (b) provide evidence that the transitions are discontinuous for $\beta = 0.5, 2.0$, the stronger evidence showing ΔC_1 as a function of n in Fig. 2(a) of the main manuscript reveals that for $\beta = 2.0$ the transition is “weakly” discontinuous, yet still shows a significant jump in C_1 for the values of n attainable in simulation.

We also measure how the number of stages visited scales with system size n for the BFW model with $\beta = 0.5$ and 2.0. Let N_k be the number of distinct stages visited up to time t_c . Fig. S4(c) shows that $N_k \sim n^{0.603}$ for $\beta = 0.5$ and $N_k \sim n^{0.292}$ for $\beta = 2.0$. This is consistent with our theoretical analysis that for $\beta = 0.5$, the stage increases slowly in subcritical region while for $\beta = 2.0$, direct growth makes the stage increases very quickly (see main manuscript). Finally, in the main manuscript, we study $n(s)$ the component

size density which starts by emphasizing the abundance of the smallest size components. Also interesting is the rank-size component distribution at t_c , which instead emphasizes the largest components. As shown in Fig. S4(d), C_i for $\beta = 0.5$ does not show scaling, yet for $\beta = 2$ we find $C_i \sim i^{-\delta}$ with $\delta = 0.62$.

S.5 Stability of multiple giant components

For any $\beta \in [0.5, \infty]$ it is always observed that two components of significant size exist at some point after the critical point, which remain separate throughout the subsequent evolution. Furthermore we find $C_1 > 0.5$ and asymptotically $C_2 = 1 - C_1$. Similar to the mechanism shown in [3] the stability of simultaneous giants is due to the high probability of sampling edges internal to the giant components in the supercritical regime. Once $C_1 > 0.5$ (as we determine numerically is the case), the probability of sampling a random edge which leads to a component of size larger than stage k is at most $P = 2C_1(1 - C_1) < 1/2$. Thus the fraction of accepted edges over total sampled edges increases to some positive value larger than $1/2$ and k stops increasing, meaning to two giant components remain distinct throughout the subsequent evolution. For more details see [3].

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